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Random Walks on a Lattice

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This paper discusses the mean-square displacement for a random walk on a two-dimensional lattice, whose transitions to nearest-neighbor sites are symmetric in the horizontal and vertical directions and depend on the column currently occupied. Under a uniform density condition for the step probabilities it is shown that the horizontal mean-square displacement after n steps is asymptotically proportional to n, and independent of the particular column configuration. The model generalizes that of Seshadri, Lindenberg, and Shuler and the arguments are essentially probabilistic.

KEY WORDS: Random walk; random lattice; periodic lattice; recurrence relations.

1. INTRODUCTION

In a recent paper,[1] there is a detailed study of a random walk on a class of two-dimensional lattices, namely, those with two types of columns called "strong" and "weak" which have different scattering characteristics. Properties of the walk investigated included the mean-square displacements of the horizontal and vertical components of the walk, the probability of return to the origin, and number of distinct lattice sites visited.

All these properties were successfully analyzed in the case when the two column types formed a strictly periodic array. For the mean-square displacements, a proof was given for a completely general array of types with an asymptotic density of strong columns. However, there is an oversight in this proof which it is not easy to overcome; further elucidation is given in Section 3. Section 4 contains a different approach to the problem, which requires only simple probability theory and no Fourier transforms, matrix algebra, or Tauberian theorems. The result proved there

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permits any number of column types and generalizes the strictly periodic case although it still imposes a fairly stringent restriction on the column array; roughly, it requires that an asymptotic density exists *uniformly* over the array. This approach gives a particularly elementary proof for the two-type periodic case. The paper concludes with discussion in Section 5.

2. THE MODEL

We consider a slight generalization of the model in Ref. 1, in which a random walk at a lattice site on column j moves with probability p_j to either horizontal neighbor and with probability $\frac{1}{2} - p_j$ to either vertical neighbor at each step. Thus the transition mechanism of the walk depends only on which column the walker is presently located. Because of the symmetry, it is easy to see, as in Ref. 1, that the mean-square displacement after n steps is n. Thus we consider in detail only horizontal movements, described more formally as follows.

Let $\mathsf{X} \equiv \{X_n\}$ $(n = 0, 1, \ldots)$, $X_0 = 0$, be a symmetric random walk on the integers $\mathbb{Z}$. Thus X_n is an integer-valued random variable representing the position of X after n steps. The transition probabilities are

$$\mathrm{Prob}(X_{n+1} = j \pm 1 \mid X_n = j) = p_j$$
$$\mathrm{Prob}(X_{n+1} = j \mid X_n = j) = 1 - 2p_j$$

for $j \in \mathbb{Z}$ and $n = 0, 1, \ldots$.

For any positive integer M, let $\mathsf{E} \equiv \{rM, r \in \mathbb{Z}\}$. Clearly, X proceeds via a series of increments between its successive visits to E. Let

$$T_i = \text{step-number of } i\text{th visit to } \mathsf{E} \text{ by } \mathsf{X}\ (i = 1, 2, \ldots), \qquad T_0 = 0$$
$$\xi_i = X_{T_i}\ (i = 1, 2, \ldots), \qquad \xi_0 = 0$$
$$\tau_i = T_i - T_{i-1}\ (i = 1, 2, \ldots)$$
$$\nu_n = \max\{\nu : T_\nu \leqslant n\}\ (n = 0, 1, 2, \ldots)$$
$$Y_i = \xi_i - \xi_{i-1}\ (i = 1, 2, \ldots)$$

Note that Y_i takes only values $-M, 0, M$.

The condition we impose on the p_j's is as follows. For some $0 < \gamma < \infty$,

$$(U) \quad \lim_{l \to \infty} \sup_{r \in \mathbb{Z}} \left| \frac{1}{l} \sum_{j=r}^{r+l} \frac{1}{p_j} - \gamma \right| = 0$$

Note that this is certainly satisfied for any strictly periodic array of columns, for which there exists a positive integer Q such that

$$p_j = p_{j+Q}, \qquad j \in \mathbb{Z}$$

3. PREVIOUS WORK

As mentioned in the Introduction the case of two column types, strong and weak, is treated in Ref. 1. For $\langle X_n^2 \rangle$, there is a result analogous to our theorem except that (effectively) (U) is replaced by a much weaker condition not containing the sup term. The method of proof is to take an associated periodic lattice, with periodic continuation of column $0, \ldots, N-1$, prove the theorem for each N and then let $N \to \infty$.

Unfortunately, this program involves a change of order of limiting operations. Such manipulation is not in general valid and requires careful justification in any particular case; typically, a version of uniform convergence needs to be invoked. The point of difficulty in Ref. 1 is Eqs. (B14) and (B15), where it is not shown that the $O(\Lambda^0)$ term in (B14) is still $O(\Lambda^0)$ in (B15) after multiplication by N and $N \to \infty$.

It seems to be very difficult to resolve whether the interchange is in fact valid. Consequently, we present a different and, we believe, simpler approach to the problem under the more restrictive hypothesis (U), which still covers the periodic case that occupies nearly all the analysis in Ref. 1.

4. RESULTS

Using three lemmas proved in the Appendix, we can establish the following theorem.

Theorem. Under condition (U),

$$\langle X_n^2 \rangle \sim \frac{2n}{\gamma}$$

Proof. Define

$$W_n = X_{T_{\nu_n+1}} = \sum_{i=1}^{\nu_n+1} Y_i$$

We first find the variance of W_n and then calculate the variance of X_n from it. Write

$$W_n = \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} Y_i I_{\{\nu_n+1 \geqslant i\}} I_{\{\xi_{i-1}=xM\}} \tag{1}$$

where I_A is the indicator function of the event A. Then

$$\begin{aligned} \langle W_n \rangle &= \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} \langle Y_i I_{\{\nu_n+1 \geqslant i\}} I_{\{\xi_{i-1}=xM\}} \rangle \\ \langle W_n^2 \rangle &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} \langle Y_i Y_j I_{\{\nu_n+1 \geqslant i\}} I_{\{\nu_n+1 \geqslant j\}} \\ &\qquad \times I_{\{\xi_{i-1}=xM\}} I_{\{\xi_{j-1}=yM\}} \rangle \end{aligned} \tag{2}$$

Now the events $\{\nu_n \geqslant i-1\}$ and $\{\xi_{i-1} = xM\}$ depend only on the history of X before T_{i-1}, and Y_i depends on this history only via ξ_{i-1}. Since, by Lemma 1, $\langle Y_i \mid \xi_{i-1} = xM \rangle \equiv 0$ for all i, $\langle W_n \rangle = 0$. A similar argument shows that the mean in (2) is zero if $i \neq j$ whence, by Lemma 1,

$$\langle W_n^2 \rangle = 2M \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} p_{xM} \pi(i, x, n) \tag{3}$$

where

$$\pi(i, x, n) \equiv \langle I_{\{\nu_n + 1 \geqslant i\}} I_{\{\xi_{i-1} = xM\}} \rangle$$

To calculate the sum in (3), consider

$$V_n = \sum_{i=1}^{\nu_n + 1} \tau_i \tag{4}$$

obviously $V_n > n$. Express (4) as in (1), which gives

$$\langle V_n \rangle = \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} \langle \tau_i I_{\{\nu_n + 1 \geqslant i\}} I_{\{\xi_{i-1} = xM\}} \rangle \tag{5}$$

As above, τ_i depends on the two events in (5) only via ξ_{i-1}. In general ξ_{i-1} does influence the conditional mean of τ_i (see Lemma 2). However, under (U), we may use Lemma 3 to get, from (4) and (5),

$$n < M(\gamma + \epsilon) \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} p_{xM} \pi(i, x, n) \tag{6}$$

To obtain a converse inequality to (6), define new variables

$$\tau_i^{(B)} = \begin{cases} \tau_i & \text{if } \tau_i \leqslant B \\ B & \text{if } \tau_i > B \end{cases}$$

with corresponding $\nu_n^{(B)}$, $V_n^{(B)}$; clearly $\nu_n^{(B)} \geqslant \nu_n$. Then, since the summands in (4) are now bounded,

$$n + B \geqslant \langle V_n^{(B)} \rangle = \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} \langle \tau_i^{(B)} I_{\{\nu_n^{(B)} + 1 \geqslant i\}} I_{\{\xi_{i-1} = xM\}} \rangle \tag{7}$$

For B sufficiently large,

$$\langle \tau_i^{(B)} \mid \xi_{i-1} = xM \rangle \geqslant (1 - \epsilon) \langle \tau_i \mid \xi_{i-1} = xM \rangle \tag{8}$$

By Lemma 3 and (8), (7) gives

$$\begin{aligned} n + B &\geqslant M(1-\epsilon)(\gamma - 2\epsilon) \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} p_{xM} \langle I_{\{\nu_n^{(B)} + 1 \geqslant 1\}} I_{\{\xi_{i-1} = xM\}} \rangle \\ &\geqslant M(1-\epsilon)(\gamma - 2\epsilon) \sum_{i=1}^{\infty} \sum_{x \in \mathbb{Z}} p_{xM} \pi(i, x, n) \end{aligned} \tag{9}$$

Given any $\epsilon > 0$, we therefore have from (3), (6), (8), and (9) that, for M and B sufficiently large,

$$\frac{2n}{\gamma+\epsilon} < \langle W_n^2 \rangle < \frac{2(n+B)}{(1-\epsilon)(\gamma-2\epsilon)} \tag{10}$$

Write

$$X_n = W_n - \omega_n \tag{11}$$

where

$$0 \leqslant \omega_n \leqslant M \tag{12}$$

Thus

$$\langle X_n^2 \rangle = \langle W_n^2 \rangle - 2\langle \omega_n W_n \rangle + \langle \omega_n^2 \rangle \tag{13}$$

By the Cauchy–Schwarz inequality and (12),

$$|\langle \omega_n W_n \rangle| \leqslant M\{\langle W_n^2 \rangle\}^{1/2} \tag{14}$$

Combining (10)–(14) gives

$$\frac{2}{\gamma+\epsilon} \leqslant \liminf_{n\to\infty} \left\langle \frac{X_n^2}{n} \right\rangle \leqslant \limsup_{n\to\infty} \left\langle \frac{X_n^2}{n} \right\rangle \leqslant \frac{2(1+\epsilon)}{(1-\epsilon)^2(\gamma-\epsilon)}$$

Since ϵ is arbitrary, we have

$$\lim_{n\to\infty} \left\langle \frac{X_n^2}{n} \right\rangle = \frac{2}{\gamma}$$

as required. ■

Corollary 1. If the p_j are periodic, with period Q, then

$$\langle X_n^2 \rangle \sim 2n \Bigg/ \left(\frac{1}{Q} \sum_{j=0}^{Q-1} \frac{1}{p_j} \right)$$

Corollary 2. If there are only strong or weak columns let S represent the set of strong columns. If $p_j = \phi$, ($j \in \mathrm{S}$), $p_j = \psi$ otherwise, and

$$\lim_{l\to\infty} \sup_{r\in\mathbb{Z}} \frac{1}{l} \left| \sum_{j\in[r,r+l]\cap \mathrm{S}} \frac{1}{p_j} - \alpha/\phi \right| = 0$$

then

$$\langle X_n^2 \rangle \sim 2n\left(\frac{\alpha}{\phi} + \frac{1-\alpha}{\psi} \right)$$

Proofs of these corollaries are obvious.

5. DISCUSSION

The way this paper avoids the problem mentioned in Section 3 is by the uniformity included in condition (U). In a sense, then, we have assumed the problem away! Nonetheless, (U) still seems sufficiently useful in its own right to merit the study. It covers all strictly periodic cases. It also handles the case when, say, every Mth column belongs to S and there is a set proportion of strong columns between them, not necessarily with strict periodicity. Both these situations are mentioned in Ref. 1. Note that (U) is *not* satisfied for a two-type column array generated by independent Bernoulli variables, since with probability one any such array will contain an arbitrarily long run of one type of column.

We note that the methods of this paper also provide an exceptionally simple direct proof of Corollary 1, which is a generalization of the result in Ref. 1 for any strictly periodic array of columns. For then the Y_i and τ_i are independent and identically distributed random variables, the E points are regeneration points of the process, and X_n is a cumulative process. By Theorem 8 of Smith,[2] since $\langle Y_i \rangle = 0$, we can immediately say that

$$\langle X_n^2 \rangle \sim n \langle Y_i^2 \rangle / \langle \tau_i \rangle$$

The elementary lemmas 1 and 2 easily give us the two means, since (A.3) becomes

$$\langle \tau_i \rangle = p_0 \left(\sum_{j=0}^{M-1} \frac{1}{p_j} \right) \tag{15}$$

Thus in this special case the only algebra needed is the solution of two sets of recurrence relations, in contrast to the formidable machinery of Ref. 1. This approach also shows clearly, from (15), why it is the totalities of column types and not their positions, which affect the solution.

The methods of this paper do not appear to help with the other problems treated in Ref. 1.

It has been pointed out to us that a more general result, namely, the asymptotic normality of X_n, can be proved under the weaker condition of (effectively) (U) without the supremum. The result of this paper under the weaker condition follows as a corollary. However, the methods required are of some technical complexity, whereas hopefully the approach presented here provides a simple guide to the probabilistic format for handling such problems. Note that, in the periodic case, the asymptotic normality comes directly from Theorem 9 of Smith.[2]

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APPENDIX

We prove here three simple probability lemmas.

Lemma 1. For $i = 1, 2, \ldots,$

$$\langle Y_i \mid \xi_{i-1} = xM \rangle = 0$$

$$\langle Y_i^2 \mid \xi_{i-1} = xM \rangle = 2p_{xM} M$$

Proof. Consider $i = 1$. Let

$$f_j = \Pr(\text{X reaches } M \text{ before 0 from } j) \qquad j \in (0, 1, \ldots, M)$$

Clearly,

$$\begin{aligned} f_j = p_j f_{j+1} + (1 - 2p_j) f_j + p_j f_{j-1} \qquad (j = 1, \ldots, M-1) \\ f_0 = 0, \qquad f_M = 1 \end{aligned} \tag{A.1}$$

From (A.1),

$$f_{j+1} - 2f_j + f_{j-1} = 0$$

which implies that f_j is linear in j. The coefficients are determined by the boundary conditions, giving

$$f_j = j/M.$$

Since the same result holds for reaching $-M$ before 0, starting from $j \in (-M, \ldots, -1, 0)$, we get

$$\Pr(Y_1 = \pm M) = p_0/M, \qquad \Pr(Y_1 = 0) = 1 - 2p_0/M \tag{A.2}$$

For general i, the same argument holds, after conditioning on ξ_{i-1}, the E point from which the ith increment starts, with p_{xM} replacing p_0 in (A.2). The lemma is now obvious. ■

Lemma 2. For $i = 1, 2, \ldots,$

$$\langle \tau_i \mid \xi_{i-1} = xM \rangle = 1 + \frac{p_{xM}}{M} \sum_{k=1}^{M-1} \sum_{j=1}^{k} \left(\frac{1}{p_{xM+j}} + \frac{1}{p_{xM-j}} \right) \tag{A.3}$$

Proof. Consider $i = 1$. Define, for $j = 0, 1, \ldots, M$,

$$m_j = \langle \text{number of steps to reach 0 or } M \text{ from } j \rangle$$

Clearly,

$$\begin{aligned} m_j = 1 + p_j m_{j+1} + (1 - 2p_j) m_j + p_j m_{j-1} \qquad (j = 1, \ldots, M-1) \\ m_0 = m_M = 0 \end{aligned} \tag{A.4}$$

From (A.4),

$$m_{j+1} - 2m_j + m_{j-1} = -1/p_j \tag{A.5}$$

Sum (A.5) over $j = 1, \ldots, k$ and then over $k = 1, \ldots, M-1$ to get

$$m_1 = \frac{1}{M} \sum_{k=1}^{M-1} \sum_{j=1}^{k} 1/p_j$$

Similarly, for the range $(0, -1, \ldots, -M)$,

$$m_{-1} = \frac{1}{M} \sum_{k=1}^{M-1} \sum_{j=1}^{k} 1/p_{-j}$$

Since

$$\langle \tau_1 \rangle = 1 + p_0(m_1 + m_{-1}) \tag{A.6}$$

the lemma is proved for $i = 1$. For general i, the same argument holds after conditioning, with p_{xM} replacing p_0 in (A.6). ■

Lemma 3. Under condition (U), given any $\epsilon > 0$,

$$Mp_{xM}(\gamma - 2\epsilon) < \langle \tau_i \,|\, \xi_{i-1} = xM \rangle < Mp_{xM}(\gamma + \epsilon)$$

for M sufficiently large.

Proof. Given any $\epsilon > 0$, and choosing M sufficiently large, we have

$$\frac{1}{M} \sum_{k=1}^{M-1} \sum_{j=1}^{k} \frac{1}{p_{xM+j}} < \epsilon + \frac{1}{M} \sum_{k=M_0}^{M-1} k(\gamma + \epsilon) < \epsilon + \frac{M}{2}(\gamma + \epsilon)$$

using (U), where M_0 is a constant depending on ϵ. Similarly,

$$\frac{1}{M} \sum_{k=1}^{M-1} \sum_{j=1}^{k} \frac{1}{p_{xM+j}} > \frac{M}{2}(\gamma - \epsilon)(1 - \epsilon) > \frac{M}{2}(\gamma - 2\epsilon)$$

There are identical bounds for the other sum in (A.3). Thus given any $\epsilon > 0$,

$$Mp_{xM}(\gamma - 2\epsilon) < \langle \tau_i \,|\, \xi_{i-1} = xM \rangle < Mp_{xM}(\gamma + \epsilon)$$

for all sufficiently large M, as required.

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